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A Review on Drinfeld modular forms of Hecke operators

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ABSTRACT— In this newspaper we determine the explicit composition of the semi simple part of the Hecke algebra that works on Drinfeld modular kinds of full level modulo T. We show that modulo T the Hecke algebra has a non-zero semi simple part. On the other hand, a well-known theorem of Serre asserts that for traditional modular varieties the action of Tl for any peculiar primary l is nilpotent modulo 2. After showing the effect for Drinfeld modular varieties modulo T, we use computations of the Hecke action modulo T showing that certain forces of the Drinfeld modular form h can't be Eigen forms. Finally, we conjecture that the Hecke algebra that functions on Drinfeld modular varieties of full level is not soft for large weight, which again contrasts the traditional situation.

KEYWORDS – Drinfeld modular forms, Reduction of Drinfeld modular forms modulo T.

I. INTRODUCTION

The nilpotence of the Hecke algebra modulo 2 for the entire modular group SL2(Z) was initially seen by Serre ([13, ?5]). The full total end result has many important arithmetic applications (see Sec. 2.7 and Ch. 5 of [10]). Lately, Nicolas and Serre ([8], [9]) have motivated the order of nilpotence and the composition of the Hecke algebra modulo 2 for the entire modular group. Taguchi and ono ([11]) have analysed other places of modular varieties that the Hecke action is nilpotent modulo 2 and also have given repercussions of the nilpotence for quadratic varieties, partition functions and central worth of twisted modular L-functions. The primary goal of today's work is to review the structure of the Hecke algebra that works on Drinfeld modular varieties for GL2 (Fq[T]) and their reductions modulo T. Unlike the situation of traditional modular varieties, the Hecke eigenvalues aren't immediately related to the 'Fourier' coefficients of Drinfeld modular varieties. This makes the research of the Hecke algebra even more challenging and the properties of the Hecke algebra continue to be of your relatively mystery. Our procedure uses Serre derivatives of Drinfeld modular varieties. We completely illustrate the semisimple area of the action of Tp mod T, where T 6= p ? Fq[T], p irreducible. That is done in Theorem 3.5. The algebra of Drinfeld modular forms modulo any degree one prime is made freely over Fq by the reduced amount of the normalized cuspidal Drinfeld modular form of the tiniest weight (see [3, (12.9)]), that your traditional circumstance where parallels the algebra of modular forms modulo 2 is freely made over F2 by the reduced amount of the traditional delta function. This shows that level one primes are likely involved analogous to the primary 2. On the other hand with the full total consequence of Serre, we show that Tp mod T has a non-trivial semi simple part for many p 6= T. The end result also supports with appropriate adjustments for just about any level one perfect.

As the Hecke action modulo T is a reduced amount of the Hecke action on Drinfeld modular varieties, explicit computations of the Hecke action modulo T allow us to confirm that certain power of h (the normalized Drinfeld cuspidal form of smallest weight) can't be Eigen forms in Section 4. Powers classically of Eigen forms should never be Eigen forms, but because the 'Fourier' coefficients of Drinfeld Eigen forms aren't related to the eigenvalues this is not a immediately much longer the entire circumstance for Drinfeld modular varieties which is interesting to find out which forces of confirmed Eigen form stay Eigen forms.

Finally, we conjecture that the Hecke algebra is not smooth above the logical function field. The Hecke algebra that classically acts on the area of modular varieties of full level is? Etale (and for that reason smooth) over Q, therefore our conjecture underscores another difference between your composition of the Hecke algebra regarding classical modular varieties and in the situation of Drinfeld modular varieties.

II. NOTATION

Let p be considered a prime quantity. Let q = p^r , with r an optimistic integer. Let A be the polynomial band Fq[T], Be the group of monic polynomials within a a+, K the fraction field of your. Let? denote the Carlitz component, i.e., the Drinfeld module of list one, defined by requiring uniquely ?T (z) = T z + zq. Through the entire article ~? Will denote a set choice of an interval of the Carlitz component.

Let Mk,m denote the group of Drinfeld modular kinds of weight k, type m, for GL2(A) over K. Let t: = t(z) = e?A~ (~?z) -1 be the most common uniformizer at 'infinity', and let ta := t(az). The function ta, for a ? A+, can be extended into a t-series with coefficients in a very

$$t_a = \frac{t^{q^{\deg_T(a)}}}{1+\cdots} = t^{q^{\deg_T(a)}} + \text{higher order terms in } t \in A[[t]]. \qquad (1)$$

Every element f of $M_{k,m}$ has a t-expansion:

$$f = \sum_{n=0}^{\infty} a_n t^n, \qquad a_n \in K,$$

which is well-defined for z close to 'infinity' and such an expansion determines f uniquely. Let

$$g := 1 - (T^q - T) \sum_{a \in A_+} t_a^{q-1}, \qquad h := \sum_{a \in A_+} a^q t_a.$$

We've identified h and g not giving their t-expansions, but by giving their A-expansions (see [12] for details). Most Drinfeld modular varieties do not may actually have A-expansions, however the ones that have the benefit that one may compute the Hecke action in it in a fairly easy manner. The algebra M_k, m is made by g and h. More precisely,

$$M_{k,m} = K\langle g^i h^j : k = (q-1)i + (q+1)j,\ j \equiv m \mod (q-1)\rangle. \qquad (2)$$

Let

$$M := \bigoplus_{k,m} M_{k,m}.$$

Then

$$M = K[g,h].$$

Let MT denote the sub algebra of M which consists of Drinfeld modular forms that have t-expansion coefficients that have denominators relatively prime to T. Such forms can be reduced modulo T by reducing their coefficients modulo T. Let

$$\widetilde{M}_{k,m} := \{\widetilde{f} \in \mathbb{F}_q[[t]] : \exists f \in M_{k,m} \cap M_T \text{ s.t. } f \equiv \widetilde{f} \mod T\}.$$

We note that eg = 1, while eh = t + higher order terms in t. If we set

$$\mu := \left\lfloor \frac{k - m(q+1)}{q^2 - 1} \right\rfloor,$$ Then (2) and eg = 1 show that

$$\widetilde{M}_{k,m} = \mathbb{F}_q \left\langle \widetilde{h}^m, \widetilde{h}^{m+(q-1)}, \ldots, \widetilde{h}^{m+\mu(q-1)} \right\rangle. \tag{3}$$

Let

$$\widetilde{M} := \bigoplus_{k,m} \widetilde{M}_{k,m}.$$

We see that Mf = Fq[eh].

The ‘false’ Eisenstein series of Gekeler ([3, (8.2)]) is defined by

Hence

$$e_{\mathfrak{p}}(z) \equiv \frac{1}{\widetilde{\mathfrak{p}}} \sum_{n=0}^{d} \alpha_n z^{q^n} \equiv \sum_{n=0}^{d} \frac{\alpha_n}{\alpha_0} z^{q^n} \mod T. \tag{7}$$

Let $\underline{i} = (i_0, \ldots, i_s)$ run over the set of $(s+1)$-tuples (s arbitrary) satisfying $i_0 + \cdots + i_s = m$, $i_0 + i_1 q + \cdots + i_s q^s = n$. According to [3, (3.8)] formula (7) shows that

$$G_{n+1,\mathfrak{p}}(X) \equiv \sum_{m \leq n} \left(\sum_{\underline{i}} \binom{m}{i_0, \ldots, i_s} \left(\frac{\alpha_1}{\alpha_0}\right)^{i_1} \cdots \left(\frac{\alpha_s}{\alpha_0}\right)^{i_s} \right) X^{m+1} \mod T, \tag{8}$$

where $\binom{m}{i_0,\ldots,i_s}$ is the multinomial coefficient $m!/(i_0! \cdots i_s!)$.

Corollary 3.1.13 from [12] shows that for $1 \leq j \leq q$:

$$\mathcal{T}_{\mathfrak{p},q-1} g = \mathfrak{p}^{q-1} g, \qquad \mathcal{T}_{\mathfrak{p},j(q+1)} h^j = \mathfrak{p}^j h^j. \tag{9}$$

Let Hk,m ⊂ End(Mk,m) be the subalgebra generated by all the Hecke operators over K for the definition of Tm when m is not a prime ideal). The reader should note that just like the classical case TmTn for relatively prime m and n, but unlike the classical case TpN = T N p for all p and all positive integers. This allows us to focus our attention on Tp where p is prime. Let Hek,m be the sub algebra of End(Mfk,m) generated by the Hecke operators away from T. Let Hss k,m be the sub algebra of Hk,m generated by operators which are absolutely semi simple. Let Hess k,m be the sub algebra of Hek,m generated by absolutely semi simple elements. The algebra Hk,m is a finite-dimensional K-algebra while Hek,m is a finite-dimensional Algebra. We note that Hss k,m is a product of finite separable extensions). We are interested in properties of $\mathcal{H}^{ss}_{k,m}$ and $\widetilde{\mathcal{H}}^{ss}_{k,m}$.

III. HECKE ALGEBRA MODULO T

In this section we determine the structure of $\widetilde{\mathcal{H}}^{ss}_{k,m}$. completely (Theorem 3.5 below). We will make use of the Serre derivatives from [3, (8.5)], which we now recall.

Define

$$\Theta := \frac{1}{\tilde{\pi}}\frac{d}{dz} = -t^2\frac{d}{dt}.$$

The operator Θ does not preserve M, but the k th Serre derivative

$$\partial_k := \Theta + kE$$

does. One can show (see [3, (8.5)]) that $\partial_k : M_{k,m} \to M_{k+2,m+1}$ and that for $k = k_1 + k_2$, f_1, f_2 Drinfeld modular forms of weights k_1, k_2, respectively, we get $\partial_k(f_1 f_2) = \partial_{k_1}(f_1)f_2 + f_1\partial_{k_2}(f_2)$.

We have (paragraph following [3, (8.7)])

$$\partial_{q+1}h = 0,$$

which, together with (4), shows that Θ preserves $\widetilde{M}$. Indeed $\partial_{q+1}h = 0$ implies that

$$\Theta h \equiv -h^2 \mod T. \tag{10}$$

As in the previous section, let $\mathfrak{p}$ stand for either a prime ideal of A different from (T) or for the monic generator of such an ideal. The reduction of $\mathfrak{p}$ modulo T, i.e., the constant term of $\mathfrak{p}$, will be denoted by $\widetilde{\mathfrak{p}}$. By using (5), we get

$$\frac{d}{dz}\mathcal{T}_{\mathfrak{p},k}f = \mathfrak{p}^{k+1}f'(\mathfrak{p}z) + \frac{1}{\mathfrak{p}}\sum_{\beta\in S_{\mathfrak{p}}} f'\left(\frac{z+\beta}{\mathfrak{p}}\right),$$

which in terms of operators shows that

$$\mathfrak{p}\cdot\Theta\mathcal{T}_{\mathfrak{p},k} = \mathcal{T}_{\mathfrak{p},k+2}\Theta. \tag{11}$$

Theorem 3.2. Let p be a prime different from T. Let n be a non-negative integer.

Then
$$\mathcal{T}_{\mathfrak{p}}h^n \equiv \widetilde{\mathfrak{p}}^n h^n + \textit{lower order terms in } h \mod T. \tag{12}$$

Proof. We prove that

$$\mathcal{T}_{\mathfrak{p}} h^n \equiv \widetilde{\mathfrak{p}}^n h^n + \text{lower order terms in } h \mod T$$

by downward induction on n.

Let n be given and let p^ν be the smallest power of p bigger than or equal to n.

If $0 \leq n \leq p$, then we already remarked in (9) that

$$\mathcal{T}_{\mathfrak{p}} g = \mathfrak{p}^{q-1} g, \qquad \mathcal{T}_{\mathfrak{p}} h^n = \mathfrak{p}^n h^n \qquad (1 \leq n \leq p),$$

and therefore

$$\mathcal{T}_{\mathfrak{p}} \widetilde{g} = \mathcal{T}_{\mathfrak{p}} 1 = \widetilde{\mathfrak{p}}^{q-1} = 1, \qquad \mathcal{T}_{\mathfrak{p}} \widetilde{h^n} = \widetilde{\mathfrak{p}}^n \widetilde{h^n} \qquad (1 \leq n \leq p).$$

This proves the result for $1 \leq n \leq p$.

Suppose that $\nu > 1$, i.e., $p^{\nu-1} < n \leq p^\nu$. Assume that the result is true for n_0 in the range $1 \leq n_0 \leq p^{\nu-1}$.

If $p \mid n$, then $n = pn_0$ for n_0 between $p^{\nu-2}$ and $p^{\nu-1}$ and by the induction hypothesis

$$\mathcal{T}_{\mathfrak{p}} h^n = (\mathcal{T}_{\mathfrak{p}} h^{n_0})^p \equiv (\widetilde{\mathfrak{p}}^{n_0} h^{n_0})^p + \text{lower order terms in } h \mod T.$$

If $p \nmid n$, then write

$$\mathcal{T}_{\mathfrak{p}} h^n \equiv \epsilon_n h^n + \text{lower order terms in } h \mod T.$$

We apply $\widetilde{\mathfrak{p}}\Theta$ to both sides of this equation and use (12):

$$\begin{aligned} \mathcal{T}_{\mathfrak{p}} (\Theta h^n) &\equiv \widetilde{\mathfrak{p}} \cdot \Theta (\mathcal{T}_{\mathfrak{p}} h^n) \\ &\equiv \widetilde{\mathfrak{p}} \cdot \Theta (\epsilon_n h^n + \text{lower order terms in } h) \mod T. \end{aligned}$$

Using (10) this becomes

$$\mathcal{T}_{\mathfrak{p}}(-nh^{n+1}) \equiv -n\widetilde{\mathfrak{p}}\epsilon_n \cdot h^{n+1} + \text{lower order terms in } h \mod T.$$

As $p \nmid n$, we have

$$\mathcal{T}_{\mathfrak{p}}(h^{n+1}) \equiv \widetilde{\mathfrak{p}}\epsilon_n \cdot h^{n+1} + \text{lower order terms in } h \mod T.$$

The equation above shows that one can prove the result for n if one assumes the result for $n+1$ and $p \nmid n$. This finishes the proof as the result for $p^\nu - 1$ is deduced from the one for p^ν (which we already have deduced from the result for the range $p^{\nu-2}$ to $p^{\nu-1}$), the result for $p^\nu - 2$ is deduced from the one for $p^\nu - 1$ and so on. □

Theorem 3.5. *We have*

$$\widetilde{\mathcal{H}}^{ss}_{k,m} = \mathbb{F}_q[(\mathbb{F}_q^*)^m],$$

with the isomorphism being given by

$$(\mathcal{T}_{\mathfrak{p}} \mod T)^{ss} \mapsto \widetilde{\mathfrak{p}}^m.$$

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As mentioned in the introduction our result is in contrast with the classical case where the Hecke operators modulo 2 are nilpotent. Since for every classical modular form with integer coefficients f there exists a positive integer i with the property that Tℓ1· · · Tℓ i, f ≡ 0 mod 2, for every collection of odd primes ℓ1. . . ℓi. For more on this see pages 34-35 in [10]. These results already suggest that the Hecke algebra action on Mk, m has a different structure than in the case of classical modular forms.

IV. THE HECKE ACTION ON $M_{k, m}$

Computations when q is prime In this subsection we use the computation of the Hecke action modulo T to conclude that certain powers of h are not Eigen forms under the assumption that q is prime. For the duration of this subsection we assume that q = p. We will keep writing q, but the reader should be aware that our assumption that q is prime is needed for the proofs that we present.

Theorem 4.2. Let d be a positive integer. If p is a monic irreducible of degree d, different from T, then h^{q^d+j} is not an Eigen form for Tp for anyj = 1. . . q − 1.

Proof. h^{q^d+j} is an Eigen form for Tp, $\widetilde{h}^{q^d+j}$ is an Eigen form for Tp.

We continue to assume that q is prime. The computations above can also be used to show that h^{h+q} cannot be an Eigen form for Tp, when p has a T-term.

We will need to work with the coefficients of the t-expansions of several lowers of eh at the same time, therefore let

$$\widetilde{h}^m = \sum_{n=m}^{\infty} a_n(m)t^n, \qquad a_n(m) \in \mathbb{F}_q.$$

Theorem 4.4. *Let* $\mathfrak{p} = T^d + \alpha_{d-1}T^{d-1} + \cdots + \alpha_1 T + \alpha_0$. *For* $j \in \mathbb{Z}$, $1 \leq j \leq (q-1)$, *we have*

$$\mathcal{T}_{\mathfrak{p}} h^{q+j} \equiv \widetilde{\mathfrak{p}}^j \alpha_1 h^{j+1} + \widetilde{\mathfrak{p}}^{j+1} h^{q+j} \mod T.$$

Proof. Let

$$\mathcal{T}_{\mathfrak{p}} h^{q+j} \equiv \epsilon_{j+1} h^{j+1} + \epsilon_{q+j} h^{q+j} \mod T.$$

Theorem 3.2 shows that $\epsilon_{q+j} = \widetilde{\mathfrak{p}}^{j+1}$, therefore we need to compute ϵ_{j+1}. We will start with the case $j = 1$ and use Θ to get the other cases. To that end, consider the t^2-term in the t-expansion of $\mathcal{T}_{\mathfrak{p}} \widetilde{h}^{q+1}$. We know that the t^2-term is completely determined by

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$$\sum_{n=q+1}^{q^d+1} a_n(q+1)\cdot G_{n,\mathfrak{p}}(\mathfrak{p}t).$$

But if we look at the explicit formula for $G_{n,\mathfrak{p}}(X)$ modulo T, i.e., formula (8), and take $m=1$, we can see that

$$G_{q^i+1,\mathfrak{p}}(X)\equiv\frac{\alpha_i}{\alpha_0}X^2+\cdots\equiv\alpha_i\widetilde{\mathfrak{p}}^{-1}X^2+\cdots \mod T,$$

while the other $G_{n,\mathfrak{p}}(X)$ in the range $q+1\le n\le q^d+1$ do not have an X^2-term. Therefore

$$\mathcal{T}_{\mathfrak{p}}h^{q+1}\equiv\widetilde{\mathfrak{p}}^2t^2\sum_{i=1}^{d}a_{q^i+1}(q+1)\cdot\alpha_i\widetilde{\mathfrak{p}}^{-1}+\text{higher order terms in } t \mod T.$$

As $a_{q+1}(q+1)=1$, and $a_{q^i+1}(q+1)=0$ for $i\ge 2$, we see that

$$\mathcal{T}_{\mathfrak{p}}h^{q+1}\equiv\widetilde{\mathfrak{p}}\alpha_1t^2+\text{higher order terms in } t \mod T.$$

This proves the case $j=1$. For the other cases, just apply $\mathfrak{p}^{j-1}\Theta^{j-1}$ to

$$\mathcal{T}_{\mathfrak{p}}h^{q+1}\equiv\widetilde{\mathfrak{p}}\alpha_1h^2+\widetilde{\mathfrak{p}}^{q+1}h^{q+1} \mod T.$$

Theorem 4.4 shows that h^{h+q} is an Eigen form modulo T for p precisely when α1 = 0. Therefore

Theorem 4.5. *Let $\mathfrak{p}=T^d+\cdots+\alpha_1T+\alpha_0$, $\alpha_i\in\mathbb{F}_q$, be a monic irreducible, $\mathfrak{p}\neq T$. We have $\alpha_1=0$ if and only if h^{q+j} $(j=1,\ldots,p-1)$ is an eigenform for $\mathcal{T}_{\mathfrak{p}}$.*

In the next subsection we present several examples which show that in the case of Drinfeld modular forms Hecke operators can have inseparable minimal polynomials.

Example 4.6. Let $q=2$. Since the type is determined modulo $(q-1)$, we know that $m=0$. The following table shows the first weights k for which the minimal polynomial of $\mathcal{T}_{\mathfrak{p}}$ is not separable for all $\mathfrak{p}$ of degree ≤ 5:

k	9	13	15	16	17	18	19	21	23	24	25	26	27	28
$\dim_K M_{k,0}$	4	5	6	6	6	7	7	8	8	9	9	9	10	10

We note that for $k\ge 3$ $M_{k,0}$ has two one-dimensional Hecke invariant subspaces (the space of Eisenstein series and the space of single-cuspidal forms), therefore the space $M_{9,0}$ is actually the first space for which the Hecke action can be inseparable.

Example 4.7. If we consider modular forms of higher level, then there are examples of inseparability of the Hecke algebra even for weight 2. Indeed [4, (9.7.4)] shows that if $q = 2$, $\mathfrak{n} = T(T^2 + T + 1)$, then the Hecke action on $M_{2,1}(\Gamma_0(\mathfrak{n}))$ is not semisimple, therefore not separable. We wish to thank Ernst-Ulrich Gekeler for bringing our attention to this example.

Example 4.8. We also have examples of inseparability for level T when $q \neq 2$. For instance, [7, Prop. 19] shows that for $q \geq 3$ and $\mathfrak{p}$ a degree one prime, $\mathfrak{p} \neq T$, the Hecke action of $\mathcal{T}_{\mathfrak{p}}$ on $M_{3,m}(\Gamma(T))$ is not diagonalizable for all types m. Consequently, the Hecke algebra is not semisimple and therefore not separable. We want to thank Ralf Butenuth for bringing our attention to this example.

Because of such examples we conjecture that:

Conjecture 4.9. *Given q there exist $k \gg 0$ and m, $0 \leq m < q$, $k \equiv 2m \bmod (q-1)$, such that*

$$H^2(\mathcal{H}_{k,m}, M_{k,m}) \neq 0.$$

In particular, the conjecture implies that $\mathcal{H}_{k,m}$ is not smooth for some $k \gg 0$ and some m.

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